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Modeling displacement and deformation in a single matrix operation

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Abstract—Computer models are increasingly employed in the effort to understand geological structures. Such models must frequently accommodate simultaneous discrete displacement and distributed deformation, as in the case of a fault-bend fold structure, for example. Traditionally, vectors are used for the specification of displacements and rank-2 tensors for deformations. In this paper, an alternative method is described. Displacement and deformation in two dimensions are modelled using a single 3 × 3 matrix and a three-dimensional model is created with a 4 × 4 matrix. This works because of the correspondence between displacement in n dimensions and simple shear in $n + 1$ dimensions. The method is a special case of one commonly used for perspective rendering in computer graphics applications but is apparently unfamiliar to structural geologists.

INTRODUCTION

DURING tectonism rocks may be displaced along discrete faults in response to an applied force or penetratively deformed throughout a shear zone in response to an imposed stress. Traditionally, these different rock responses were studied independently; textbooks generally contained separate chapters on brittle and plastic deformation with little overlap. Recently, however, structural geologists have become increasing aware of the interaction of discrete and penetrative structures; for example, brittle and plastic deformation mechanisms are now known to interact in deforming rocks and many fold types are known to be intimately related to fault displacement both in time and space.

The standard approach to the mathematical modeling of structures is to use vectors to describe displacements and rank-2 tensors to describe deformations. Thus, in three dimensions, a point in a rock mass might under go a displacement $\mathbf{u}$,

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \tag{1}$$

and a deformation $\mathbf{D}$,

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \tag{2}$$

(Means 1976). It seems that this separate treatment of displacement and deformation is necessary because a series of displacements labelled a,b,c accumulate additively,

$$\mathbf{u}_{\text{total}} = \mathbf{u}_\text{c} + \mathbf{u}_\text{b} + \mathbf{u}_\text{a} \tag{3}$$

whereas the cumulative effect of deformation is multiplicative (by convention, tensor products are written sequentially from right to left),

$$\mathbf{D}_{\text{total}} = \mathbf{D}_\text{c}\mathbf{D}_\text{b}\mathbf{D}_\text{a}. \tag{4}$$

However, it is possible to write the effect of a vector and rank-2 tensor in a single matrix operator and thus improve the efficiency and presentation of mathematical models involving both discrete displacement and distributed deformation. This paper describes the basic theory of the combined transformation and includes a number of practical applications.

Two-dimensional case

In order to describe a two-dimensional combined displacement and deformation we must start by considering a three-dimensional deformation. Consider a card deck in the shape of a unit cube with cards oriented parallel to the x_1 and x_2 co-ordinate axes as in Fig. 1(a). The deck is free to undergo homogeneous simple shear deformation γ in any direction perpendicular to x_3, and so the deformed state may have components of shear parallel to the x_1 and x_2 axes, which we label γ_1 and γ_2,

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & \gamma_1 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{5}$$

Consider the front card, the one that cuts the x_3 co-ordinate axis at $\{0, 0, 1\}$. Any point $p = \{x_1, x_2, 1\}$ on this card is transformed to $p' = \{x_1', x_2', 1\}$ given by the standard deformation tensor equation,

$$p' = \mathbf{D} * p \tag{6}$$

or

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \gamma_1 \\ 0 & 1 & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \tag{7}$$

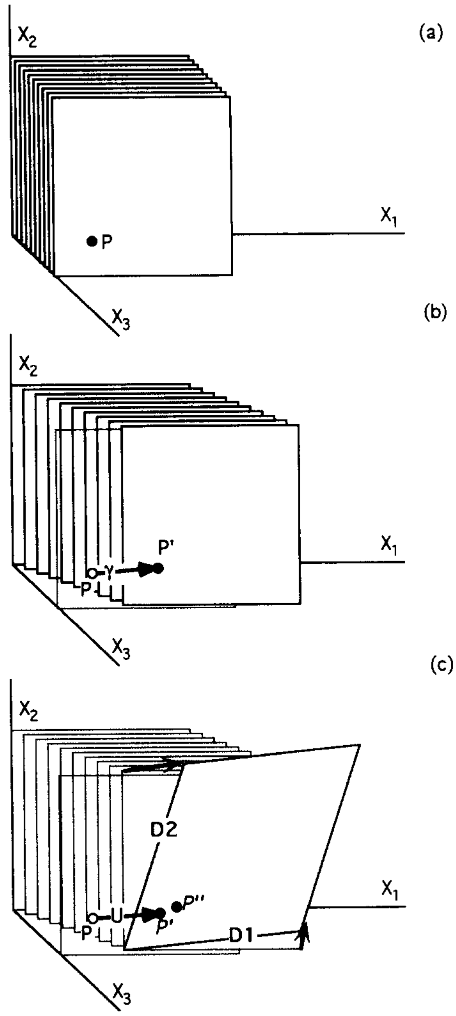

Fig. 1. (a) Card deck oriented parallel to the x_1–x_2 co-ordinate plane. p is an arbitrary point on the front card, which cuts the x_3 reference axis at $\{0, 0, 1\}$. (b) Simple shear γ indicated by bold arrow. p' is the deformed equivalent of p in (a). Dashed square indicates the initial position of the front card. (c) Deformation of the plane of the front card transforms p' to p''. Bold arrows indicate the displacement gradients. D_1 and D_2 are the column vectors of the deformation tensor $\mathbf{D}$. u is a displacement equal to γ in (b).

(Fig. 1b). If we apply two sequential phases of simple shear, denoted by subscripts a and b, the total deformation, subscripted c, is given by pre-multiplication as in equation (4). However, in this case

$$\mathbf{D}_c = \mathbf{D}_b\mathbf{D}_a \tag{8}$$

$$= \begin{bmatrix} 1 & 0 & \gamma_{b1} \\ 0 & 1 & \gamma_{b2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \gamma_{a1} \\ 0 & 1 & \gamma_{a2} \\ 0 & 0 & 1 \end{bmatrix} \tag{9}$$

$$= \begin{bmatrix} 1 & 0 & \gamma_{b1} + \gamma_{a1} \\ 0 & 1 & \gamma_{b2} + \gamma_{a2} \\ 0 & 0 & 1 \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} 1 & 0 & \gamma_{b1} \\ 0 & 1 & \gamma_{b2} \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & \gamma_{a1} \\ 0 & 1 & \gamma_{a2} \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$$= \mathbf{D}_b + \mathbf{D}_a. \tag{12}$$

Thus the cumulative deformation, equation (10), is given both by tensor multiplication, equations (8) and (9), and tensor addition, equations (11) and (12).

Now imagine that deformation is possible within the plane of the front card so that a unit square is deformed into a parallelogram by

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{21} & D_{22} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

prior to shearing. Combining equations (5) and (13) gives a general deformation

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & \gamma_1 \\ D_{21} & D_{22} & \gamma_2 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

comprising a two-dimensional deformation of the plane of the card indicated by elements $\mathbf{D}_{ij}$ $[i, j = 1, 2]$, combined with a simple shear which displaces but does not distort the card plane (Fig. 1c). Remembering that shear strain (γ) is defined as the displacement ($\mathbf{u}$) divided by the orthogonal distance from the reference frame, which is unity for the front card, equation (14) may be rewritten

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & u_1 \\ D_{21} & D_{22} & u_2 \\ 0 & 0 & 1 \end{bmatrix}. \tag{15}$$

From equation (6), the point $p = \{x_1, x_2, 1\}$ is transformed to $p' = \{x_1', x_2', 1\}$ by

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & u_1 \\ D_{21} & D_{22} & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}. \tag{16}$$

The bottom row of this equation expresses the fact that there is no change in the x_3 dimension and thus may be treated as a dummy dimension. The top two rows give the combined effect of a two-dimensional displacement $\mathbf{u}$ and a two-dimensional deformation $\mathbf{D}$ upon points in the $\{001\}$ plane. Normally, we assume $x_3 = 0$ for any two-dimensional analysis, but $x_3 = 1$ is more convenient in this case.

Three-dimensional case

A three-dimensional combination of displacement and deformation requires a 4×4 matrix,

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ 1 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{13} & u_1 \\ D_{21} & D_{22} & D_{23} & u_2 \\ D_{31} & D_{32} & D_{33} & u_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ 1 \end{bmatrix}, \tag{17}$$

where the bottom row is again a dummy dimension, $x_4 = 1$. The underlying principle is the correspondence between displacement $\mathbf{u}$ in n dimensions and simple shear γ in $n + 1$ dimensions. The matrix incorporating the extra dimension represents a homogeneous deformation and satisfies the tensor transformation rule for rank-2 tensors. Of course, in four dimensions, the deformation tensor is a mathematical concept only, since it is impossible to visualize simple shear of a hypercube leaving one of its 'sectional' cubes undeformed!

Although derived independently here using concepts of shear and stretch familiar to geologists, equation (16) is but a special case of a general homogeneous coordinate transformation commonly used in computer graphics applications (e.g. Foley *et al.* 1990), especially for the display of perspective views (in which case, variables may appear in the bottom row). However, the approach is unfamiliar to structural geologists in general.

Displacement and displacement gradients

The displacement gradients tensor **u** is simply related to the deformation (alias position gradients) tensor **D** (Ramsay & Huber 1983) so we may write an equation that combines the effects of rigid displacements and displacement gradients,

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x_1'}{\partial x_1} & \dfrac{\partial x_1'}{\partial x_2} & u_1 \\ \dfrac{\partial x_2'}{\partial x_1} & \dfrac{\partial x_2'}{\partial x_2} & u_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}$$

$$- \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \tag{18}$$

$$= \begin{bmatrix} \dfrac{\partial u_1}{\partial x_1} & \dfrac{\partial u_1}{\partial x_2} & u_1 \\ \dfrac{\partial u_2}{\partial x_1} & \dfrac{\partial u_2}{\partial x_2} & u_2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}. \tag{19}$$

If the net displacements of a set of marker points are known, then the displacement field may be factored into a common, discrete displacement and a residual displacement gradient, from which the deformation state follows.

PRACTICAL EXAMPLES

(1) *Rigid rotation about a point other than the origin*

It is well known that a rotation through an angle θ about the origin is given by

$$\mathbf{R} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{20}$$

However, if the center of rotation is at $C = \{C_1, C_2\}$, then the final position $p' = \{x_1', x_2'\}$ of a point initially at $p = \{x_1, x_2\}$ is given by translating C to the origin, applying the rotation R, and then restoring the center C (Fig. 2). In the matrix notation outlined here these operations are,

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \\ 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} 0 & 1 & 0 & -C_1 \\ 0 & 0 & 1 & -C_2 \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix} \tag{21}$$

or

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & C_1(1-\cos\theta) + C_2\sin\theta \\ \sin\theta & \cos\theta & C_2(1-\cos\theta) + C_1\sin\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 1 \end{bmatrix}. \tag{22}$$

Equation (22) may be written in alternative form, using θ as the independent variable,

$$\begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - C_1 & C_2 - x_2 & C_1 \\ x_2 - C_2 & C_1 - x_1 & C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \\ 1 \end{bmatrix} \tag{23}$$

which is convenient when modeling the effects of various rotations on a given initial point.

(2) *Fault-related flexure*

Consider a rock mass undergoing simple shear deformation during displacement over a fault bend (Fig. 3a). Let u_1 be an increment of the fault displacement along the lower flat and let the dip of the ramp be 2ψ. The deformation associated with the ramp may be viewed as a dextral simple shear

$$2\gamma = 2\tan\psi \tag{24}$$

combined with a counterclockwise rotation

$$\theta = -2\psi \tag{25}$$

about a center of rotation $C = \{C_1, C_2\}$ located at the foot of the ramp. The displacement of a point p to p' along the lower flat is simply given by

$$\mathbf{p}' = \mathbf{p} + \mathbf{u}_1. \tag{26}$$

If p' now lies within the kinked domain, then a deformation must be applied bring it to p'' (Fig. 3b) and then p''' (Fig. 3c) using the tensor product

$$\begin{bmatrix} x_1''' \\ x_2''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 2\gamma & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -C_1 \\ 0 & 1 & -C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} \tag{27}$$

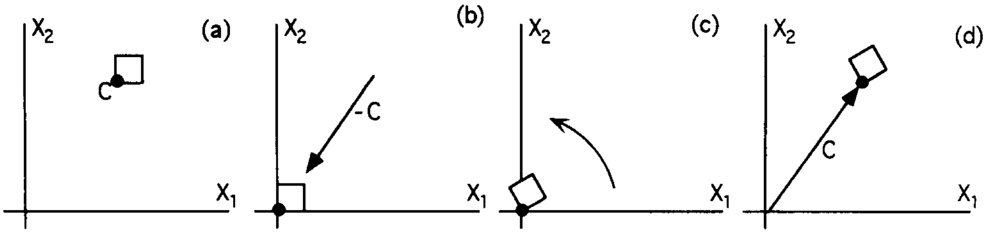

Fig. 2. Rotation of an elemental cube about a centre C. (a) Initial position. (b) Translation through −C. (c) Rotation about the co-ordinate origin. (d) Translation by +C.

or

$$\begin{bmatrix} x_1''' \\ x_2''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & 2\gamma\cos\theta - \sin\theta & 0 \\ \sin\theta & 2\gamma\sin\theta + \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & -C_1 \\ 0 & 1 & -C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix} \tag{28}$$

which represents a translation of the center of rotation C to the origin, application of shear strain parallel to the x_1 axis, rotation through the ramp angle and restoration of the center of rotation. For computational efficiency, this product may be reduced to a single 3×3 matrix prior to transformation of a large set of points,

$$\begin{bmatrix} x_1''' \\ x_2''' \\ 1 \end{bmatrix} =$$

$$\begin{bmatrix} \cos\theta & \gamma\cos\theta - \sin\theta & C_1 + C_2\sin\theta - (C_1 + \gamma C_2)\cos\theta \\ \sin\theta & \gamma\sin\theta + \cos\theta & C_2 - C_2\cos\theta - (C_1 + \gamma C_2)\sin\theta \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix}. \tag{29}$$

Alternatively, the transformation may be viewed as a sinistral simple shear -2γ oriented parallel to the kink plane, bringing point p' to p''' (Fig. 3d),

$$\begin{bmatrix} x_1''' \\ x_2''' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & C_1 \\ 0 & 1 & C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 \\ -2\gamma & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & -C_1 \\ 0 & 1 & -C_2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1' \\ x_2' \\ 1 \end{bmatrix}. \tag{30}$$

The net effect is the same because the distortion produced by any 'real' dextral simple shear parallel to one circular section of the strain ellipsoid (the fault flat) is equivalent to a 'virtual' sinistral simple shear parallel to the second circular section (the kink plane).

(3) *Faulted folds*

The final example demonstrates the application of the method to a heterogeneously strained domain as represented by the faulted folds in Fig. 4. A set of faults

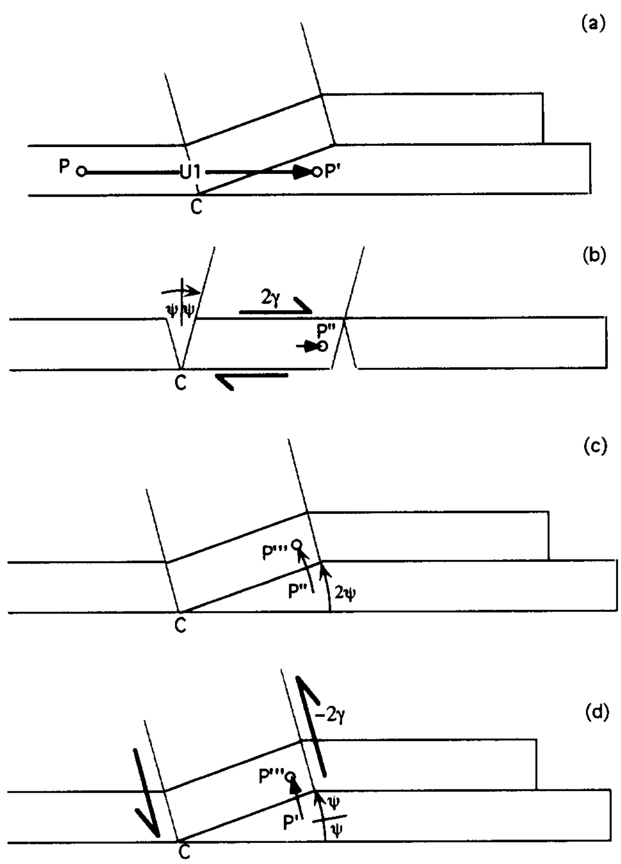

Fig. 3. Displacement and deformation associated with a fault bend fold. (a) Initial point p would be displaced by $\mathbf{u}_1$ to p' in the absence of the fault. Since p' lies in the kink zone, a correction is necessary. (b) Dextral simple shear 2γ parallel to the fault flat. ψ is the angular shear for half the total deformation. Dashed lines are circular sections of the strain ellipsoid. Light arrow shows the rotation of the inclined circular section. Point p' in (a) is displaced to p'' (bolt arrow). (c) Counterclockwise rotation (light arrow) through 2ψ about the center of rotation C located at the base of the fault ramp. Point p'' in (b) is rotated to the final position p'''. (d) Alternative to steps (b) and (c). Sinstral simple shear (bold half arrows) of -2γ oriented parallel to the kink fold axes. Point p' in (a) is displaced directly (bold arrow) to p'''. The ramp angle is 2ψ (light arrow) and the flat and ramp mark the initial and final orientations of one circular section of the strain ellipsoid (the second circular section is fixed parallel to the kink axes).

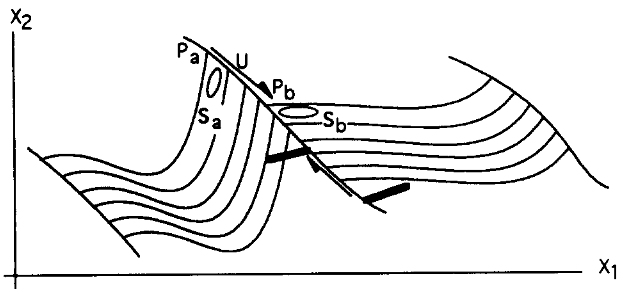

Fig. 4. Non-rigid displacement of previously deformed rocks along a curved fault. Arrows indicate fault displacement. Thick black line is a deflected dike. Ellipses represent stretch at previously juxtaposed locations p_a and p_b. u is the net displacement, $p_b - p_a$.

have been mapped through a sequence of folded rocks containing strain markers such as deformed pebbles or oolites. Two localities specified by position vectors $\mathbf{p}_a$ and $\mathbf{p}_b$ were originally juxtaposed but are now separated by a net fault displacement $\mathbf{u}$

$$\mathbf{u} = \mathbf{p}_a - \mathbf{p}_b. \tag{31}$$

Strain measurements in a common $\{x_1,x_2\}$ reference frame yield two stretch (irrotational deformation) tensors, $\mathbf{S}_a$ and $\mathbf{S}_b$. In this case we may write

$$\begin{bmatrix} Sb_{11} & Sb_{12} & Pb_1 \\ Sb_{12} & Sb_{22} & Pb_2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & u_1 \\ S_{12} & S_{22} & u_2 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} Sa_{11} & Sa_{12} & Pa_1 \\ Sa_{12} & Sa_{22} & Pa_2 \\ 0 & 0 & 1 \end{bmatrix}, \tag{32}$$

where $\mathbf{S}$ is the stretch of sidewall b *relative to* sidewall a resulting from non-rigid faulting. Whilst a single equation of this type cannot be solved for all unknown components, there are realistic situations in which a combination of strain markers, displaced veins or dikes (thick line in Fig. 4), and fault strike data, may yield at least a partial picture of the displacement and deformation fields surrounding the fault. Conversely, when $\mathbf{S}$ is independently known or inferred from a fault mechanism model, or where rigid displacement, i.e. $\mathbf{S} = 1$, on a straight fault is assumed, the amount of displacement $\mathbf{u}$ may be obtained by determining the strain compatibility of formerly juxtaposed domains. Values of displacement components $\{u_1, u_2\}$ may be iteratively incremented and associated stretch measurements, $\mathbf{S}_a$, $\mathbf{S}_b$ tested for compatibility across the interface.

CONCLUSION

Equations (27)–(32) illustrate the power of the method. A complex transformation may be built up from simple, understandable, steps whilst the final matrix reduces to the minimum number of computational steps—an important consideration when modeling the displacement and deformation of large sets of points.

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